Isomorphism in Euclidean and Sequence Spaces

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Contents

1 Definitions

1.1 Isomorphism in Normed Spaces

Let X and Y be any two normed spaces. An ***isomorphism*** from X to Y is defined as a bijective linear transformation from X to Y such that $||x||_X = ||T(x)||_Y$, provided that such a bijection exists. Whenever an isomorphism from X to Y exists, we say that X and Y are ***isomorphic***. For convenience, we write $||x||_X = ||T(x)||_Y$ as $||x|| = ||T(x)||$. It must be understood that the two norms are, in general, different.

1.2 Signum Function

Let $z \in \mathbb{C}$. Define sgn : $\mathbb{C} \to \mathbb{C}$ as $\text{sgn}(z) = \dfrac{z}{|z|}$ for $z \neq 0$ and $\text{sgn}(0) = 0$. Then, sgn defined as the signum function on $\mathbb{C}$.

1.3 Schauder Basis

Let X be a normed space. Let $(e_k)_{k=1}^{\infty}$ be a sequence in X such that to every $x \in X$, there corresponds a unique sequence of scalars $(x_k)_{k=1}^{\infty}$ such that $x = \sum_{k=1}^{\infty} x_k e_k$. In such a case, we say that $(e_k)_{k=1}^{\infty}$ is a Schauder basis for X.

2 Some Results of Linear Transformations

2.1 Remark

This section shall deal with some basic yet extremely useful properties of linear transformations (i.e. linear operators).

2.2 Theorems

Theorem 1. *Let X and Y be finite dimensional vector spaces. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis for X. Let $f : X \to Y$ and $g : X \to Y$ be two linear transformations such that $f(e_i) = g(e_i)$ for all $i = 1, 2, \cdots, n$. Then, $f = g$.*

Proof. Let $x \in X$ and write $x = \sum_{k=1}^{n} x_k e_k$. As f is linear and as $f(e_i) = g(e_i)$ for all $i = 1, 2, \cdots, n$, we have,

$$f(x) = \sum_{k=1}^{n} x_k f(e_k) = \sum_{k=1}^{n} x_k g(e_k) = g(x).$$

But, $x \in X$ was arbitrary. Hence, $f = g$. □

Theorem 2. *Let $(e_n)_{n=1}^{\infty}$ be a Schauder basis for a normed space X. Let $f : X \to Y$ be a bounded linear operator. Then, for $x = \sum_{k=1}^{\infty} x_k e_k$, we have, $f(x) = \sum_{k=1}^{\infty} x_k f(e_k)$.*

Proof. Let the given conditions hold. As f is linear and bounded, it is continuous. Now, for any $n \in \mathbb{N}$, we have,

$$f\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} x_k f(e_k)$$

which gives,

$$\lim_{n\to\infty} f\left(\sum_{k=1}^{n} x_k e_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} x_k f(e_k).$$

Due to continuity of f, this ultimately gives,

$$f(x)=\sum_{k=1}^{\infty}x_kf(e_k).$$

Hence the proof is complete. $\square$

Theorem 3. *Let $(e_n)_{n=1}^{\infty}$ be a Schauder basis for a normed space X. Let $f : X \to Y$ and $g : X \to Y$ be bounded linear operators such that $f(e_i) = g(e_i)$ for all $i \in \mathbb{N}$. Then, $f = g$.*

Proof. Let $x=\sum_{k=1}^{\infty}x_ke_k$. As per the given conditions and Th.2,

$$f(x)=\sum_{k=1}^{n}x_kf(e_k)=\sum_{k=1}^{n}x_kg(e_k)=g(x).$$

But, $x \in X$ was arbitrary. Hence, $f = g$. $\square$

3 Some Results on Euclidean and Sequence Spaces

3.1 Remark

In this section, we shall prove some results related to sequence spaces which shall be used in theorems of sec.4.

3.2 Theorems

Theorem 4. *Let $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. Then, $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x)=\sum_{k=1}^{n}x_ky_k$ is a linear functional such that $y_j = f(e_j)$ for every $j = 1, 2, \cdots, n$. Here, $x = (x_1, x_2, \cdots, x_n)$ and $\{e_1, e_2, \cdots, e_n\}$ is the standard basis for $\mathbb{R}^n$.*

Proof. As y is fixed and as the components of any given point in $\mathbb{R}^n$ is unique, each of the products, x_ky_k, in the given sum must unique and due to this, the sum is unique. Hence f is well defined. As the range is a subset of $\mathbb{R}$, it is a functional. Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be any two points in the given Euclidean space and let α and β be scalars. Then,

$$\begin{aligned}
f(\alpha a+\beta b) &= \sum_{k=1}^{n}(\alpha a_k+\beta b_k)y_k \\
&= \sum_{k=1}^{n}(\alpha a_ky_k+\beta b_ky_k) \\
&= \alpha\sum_{k=1}^{n}a_ky_k+\beta\sum_{k=1}^{n}b_ky_k \\
&= \alpha f(a)+\beta f(b).
\end{aligned}$$

This proves the linearity. We know that for e_j, where $j = 1, 2, \cdots, n$, the j^{th} component is 1 and the rest are zero. So, taking $x = e_j$, $f(e_j) = 1 \cdot y_j = y_j$. This completes the proof. $\square$

Theorem 5. *Let $(\vec{e}_k)_{k=1}^{\infty}$ be a sequence defined by $\vec{e}_k = (\delta_{jk})_{j=1}^{\infty}$, where the delta represents the Kronecker delta. Then, $(\vec{e}_k)_{k=1}^{\infty}$ is Schauder basis for every l_p space where, $p \neq \infty$.*

Proof. Let $\vec{x} = (x_k)_{k=1}^{\infty}$ be any sequence in l_p space, where $p \neq \infty$. We note that each $\vec{e}_k$ is in the l_p space since $\sum_{j=1}^{\infty} |\delta_{jk}|^p = 1 < \infty$, for every $\vec{e}_k$. We claim that $\vec{x} = \sum_{k=1}^{\infty} x_k \vec{e}_k$. Let us define $\vec{S}_n = \sum_{k=1}^{n} x_k \vec{e}_k$. Put $\vec{S} = \vec{x}$. Then, we have,

$$\vec{S}_n = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$$

$$\vec{S} = (x_1, x_2, x_3, \cdots).$$

Now, $\vec{S}_n - \vec{S} = (0, 0, \cdots, 0, -x_{n+1}, -x_{n+2}, \cdots)$ which is clearly in l_p space. Furthermore,

$$||\vec{S}_n - \vec{S}|| = \left(\sum_{k=n+1}^{\infty} |x_k|^p \right)^{1/p} \to 0 \text{ as } n \to \infty$$

since $\sum_{k=n+1}^{\infty} |x_k|^p$ is the remainder of the convergent series $\sum_{k=1}^{\infty} |x_k|^p$. Hence we have, $\vec{x} = \sum_{k=1}^{\infty} x_k \vec{e}_k$. In order to show that this representation is unique, we note that for $\vec{0} = (0, 0, \cdots)$, the representation is $\vec{0} = \sum_{k=1}^{\infty} 0 \cdot \vec{e}_k$ and this is the only representation for $\vec{0}$, otherwise, the sum on the right-hand side of $\vec{0} = \sum_{k=1}^{\infty} x_k \vec{e}_k$ shall be non-zero, for at least one $x_k \neq 0$. So the representation for $\vec{0}$ is unique. If possible, let, for $\vec{x} \neq 0$, $\vec{x} = \sum_{k=1}^{\infty} x_k \vec{e}_k = \sum_{k=1}^{\infty} x'_k \vec{e}_k$ and $x_{k_0} \neq x'_{k_0}$ for some $k = k_0$. This clearly gives, due to the convergence of series, $\sum_{k=1}^{\infty} (x_k - x'_k)\vec{e}_k = \vec{0}$ and so, $x_k = x'_k$ for all $k \in \mathbb{N}$. So $x_{k_0} = x'_{k_0}$, which, along with our initial assumption about these scalars, gives a contradiction. Hence, the representation is unique and hence, $(\vec{e}_k)_{k=1}^{\infty}$ is a Schauder basis for l_p spaces. □

Theorem 6. *Let $p, q > 1$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let $\vec{y} = (y_n)_{n=1}^{\infty} \in l_p$. Then, $f : l_p \to \mathbb{K}$ defined by $f(\vec{x}) = \sum_{k=1}^{\infty} x_k y_k$ is a bounded linear functional such that $y_j = f(\vec{e}_j)$ for every $j \in \mathbb{N}$. Here, $\vec{x} = (x_n)_{k=1}^{\infty} \in l_p$ and $(\vec{e}_k)_{k=1}^{\infty}$ is as in Th.5.*

Proof. Under the given conditions, we can observe that

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{\infty} |y_k|^q \right)^{1/q} < \infty$$

due to Hölder's inequality. So, the sum is in $\mathbb{K}$ due to its absolute convergence and, due to uniqueness of each x_k, we shall conclude that f is well defined. Let $\alpha, \beta \in \mathbb{K}$ and $\vec{s} = (s_n)_{n=1}^{\infty}, \vec{t} = (t_n)_{n=1}^{\infty} \in l_1$.Then,

$$\begin{aligned} f\left(\alpha\vec{s} + \beta\vec{t}\right) &= \sum_{k=1}^{\infty}(\alpha s_k + \beta t_k)y_k \\ &= \alpha\sum_{k=1}^{\infty} s_k y_k + \beta\sum_{k=1}^{\infty} t_k y_k \\ &= \alpha f(\vec{s}) + \beta f(\vec{t}). \end{aligned}$$

This shows linearity and from the inequality we obtained, we can write that

$$|f(\vec{x})| = \left|\sum_{k=1}^{\infty} x_k y_k\right| \leq \sum_{k=1}^{\infty} |x_k y_k| \leq ||\vec{x}||\ ||\vec{y}||.$$

This shows f is bounded. By definition of $\vec{e}_k$, $f(\vec{e}_k) = 1 \cdot y_k = y_k$. This completes the proof. □

Theorem 7. *Let $\vec{y} = (y_n)_{n=1}^{\infty} \in l_\infty$. Then, $f : l_1 \to \mathbb{K}$ defined by $f(\vec{x}) = \sum_{k=1}^{\infty} x_k y_k$ is a bounded linear functional such that $y_j = f(\vec{e}_j)$ for every $j \in \mathbb{N}$. Here, $\vec{x} = (x_n)_{k=1}^{\infty} \in l_1$ and $(\vec{e}_k)_{k=1}^{\infty}$ is as in Th.5.*

Proof. We know that, $\sum_{k=1}^{\infty} |x_k| < \infty$ and as $\vec{y} \in l_\infty$, it is bounded. So,

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| &= \sum_{k=1}^{\infty} |x_k|\ |y_k| \\ &\leq \sup_{k\in\mathbb{N}} |y_k| \sum_{k=1}^{\infty} |x_k| < \infty. \end{aligned}$$

As each of the terms, x_k, of $\vec{x}$ are unique, $\vec{y}$ is fixed and as per the above absolute convergence, it follows that the sum in the definition of f converges and is unique. Hence, f is well defined. Let $\alpha, \beta \in \mathbb{K}$ and $\vec{s} = (s_n)_{n=1}^{\infty}, \vec{t} = (t_n)_{n=1}^{\infty} \in l_1$.Then,

$$\begin{aligned} f\left(\alpha\vec{s} + \beta\vec{t}\right) &= \sum_{k=1}^{\infty}(\alpha s_k + \beta t_k)y_k \\ &= \alpha\sum_{k=1}^{\infty} s_k y_k + \beta\sum_{k=1}^{\infty} t_k y_k \\ &= \alpha f(\vec{s}) + \beta f(\vec{t}). \end{aligned}$$

This shows linearity and from the inequality we obtained, we can write that

$$|f(\vec{x})| = \left|\sum_{k=1}^{\infty} x_k y_k\right| \leq \sum_{k=1}^{\infty} |x_k y_k| \leq ||\vec{x}||\ ||\vec{y}||.$$

This shows f is bounded. By definition of $\vec{e}_k$, $f(\vec{e}_k) = 1 \cdot y_k = y_k$. This completes the proof. □

Theorem 8. *Let $\vec{e}_0 = (1)_{n=1}^{\infty}$ and let $\vec{e}_k$ be defined as in Th.5. Let c denote the space of all convergent sequences on $\mathbb{C}$. Then, $(\vec{e}_k)_{k=0}^{\infty}$ forms a Schauder basis for c and, each $\vec{x} = (x_n)_{n=1}^{\infty} \in c$, converging to l, has a representation*

$$\vec{x} = l\vec{e}_0 + \sum_{k=1}^{\infty}(x_k - l)\vec{e}_k.$$

Proof. Under the given conditions, let us write $\vec{S} = \vec{x} - l\vec{e}_0 = (x_1 - l, x_2 - l, \cdots)$ and $\vec{S}_n = \sum_{k=1}^{n}(x_k - l)\vec{e}_k = (x_1 - l, x_2 - l, \cdots, x_n - l, 0, 0, \cdots)$. This gives,

$$\vec{S} - \vec{S}_n = (0, 0, \cdots, 0, x_{n+1} - l, x_{n+2} - l, \cdots).$$

So, we must have,

$$\begin{aligned}
||\vec{S}_n - \vec{S}|| &= \sup_{n\in\mathbb{N}}\{0, |x_{n+1} - l|, |x_{n+2} - l|, \cdots\} \\
&= \sup\{|x_2 - l|, |x_3 - l|, \cdots, |x_3 - l|, |x_4 - l|, \cdots, |x_4 - l|, \cdots\} \\
&= \sup\{|x_2 - l|, |x_3 - l|, \cdots\} \\
&= \sup_{n\in\mathbb{N}}|x_{n+1} - l|
\end{aligned}$$

under the assumption that the norm is not zero, since in that case, we have $||\vec{S}_n - \vec{S}|| \to 0$ as $n \to \infty$, trivially. Now, we know that x_n converges to l. So, take any arbitrary $\varepsilon > 0$ and let $n \in \mathbb{N}$ be a natural number that corresponds to our $\varepsilon > 0$. Furthermore assume that $n \geq N$. Then, due to convergence, under these conditions, $|x_n - l| < \frac{\varepsilon}{2}$. As $n + 1 \geq n \geq N$ (under our assumptions), we can say that they implicate $|x_{n+1} - l| < \frac{\varepsilon}{2}$. This means that $\sup_{n\geq N}|x_{n+1} - l| \leq \frac{\varepsilon}{2} < \varepsilon$. Hence, under the assumption that $n \geq N$, we have,

$$||\vec{S}_n - \vec{S}|| = \sup_{n\in\mathbb{N};\, n\geq N}|x_{n+1} - l| = \sup_{n\geq N}|x_{n+1} - l| < \varepsilon.$$

This shows $\vec{S}_n \to \vec{S}$ and hence, we get $\vec{x} - l\vec{e}_0 = \sum_{k=1}^{\infty}(x_k - l)\vec{e}_k$. This simply means

$$\vec{x} = l\vec{e}_0 + \sum_{k=1}^{\infty}(x_k - l)\vec{e}_k.$$

To show that representation is unique, we must show all the scalars in the above representation are unique. For this, consider $\vec{0} = (0)_{n=1}^{\infty}$, which converges to $l = 0$. So, we must have its representation as

$$\vec{0} = 0 \cdot \vec{e}_0 + \sum_{k=1}^{\infty} x_k\vec{e}_k = \sum_{k=1}^{\infty} x_k\vec{e}_k.$$

As in Th.5, we may have all of $x_k = 0$, and none of them can take any other value. So, $\vec{0} = 0 \cdot \vec{e}_0$. If we alter the coefficient of $\vec{e}_0$, we no longer get $\vec{0}$. So, the coefficient of each of $\vec{e}_k$ are unique and

zero for the null sequence. Let $\vec{x} = l'\vec{e}_0 + \sum_{k=1}^{\infty}(x_k' - l')\vec{e_k}$ be any other representation for $\vec{x}$. Then,

$$(l - l')\vec{e}_0 + \sum_{k=1}^{\infty}(x_k' - l')\vec{e_k} = \vec{0}$$

and hence, $l = l'$ and also, every $x_k = x_k'$. This shows the uniqueness of representation and hence, $(\vec{e}_k)_{k=0}^{\infty}$ forms a Schauder basis for c. □

Theorem 9. *Let $\vec{y} = (y_k)_{k=0}^{\infty} \in l_1$. Let $\vec{x} = (x_n)_{n=1}^{\infty} \in c$ with $x_n \to l$. Define $\Phi : c \to \mathbb{K}$ by*

$$\Phi(\vec{x}) = y_0 \cdot l + \sum_{k=1}^{\infty} x_k y_k.$$

Then, Φ is a bounded linear functional on c such that $\Phi(\vec{e}_k) = y_k$ for all $k \in \mathbb{N}$ and $\Phi(\vec{e}_0) - \sum_{k=1}^{\infty}\Phi(\vec{e}_k) = y_0$. Here $\vec{e}_k$ for $k \in \mathbb{N}$ are as in Th.5.

Proof. Under the given conditions, we may observe that since a convergent complex sequence is bounded,

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \sup_{k\in\mathbb{N}} |x_k| \sum_{k=1}^{\infty} |y_k| < \infty.$$

So, the sum in the definition of Φ is convergent and due to uniqueness of terms of x_k, the sum given by the function is unique. So, Φ is well defined. To show that Φ is linear, let $\alpha, \beta \in \mathbb{K}$ and $\vec{s} = (s_k)_{k=0}^{\infty}$, $\vec{t} = (t_k)_{k=0}^{\infty} \in c$. Then $\alpha s_n + \beta t_n \to \alpha s + \beta t$, where, $s_n \to s$ and $t_n \to t$. Now, $y_0(\alpha s + \beta t) = y_0\alpha s + y_0\beta t$. Also, $\sum_{n=1}^{\infty}(\alpha s_n + \beta t_n)y_k = \alpha\sum_{k=1}^{\infty} s_k y_k + \beta\sum_{k=1}^{\infty} t_k y_k$. Thus, we may obtain,

$$\begin{aligned}
\Phi(\alpha\vec{s} + \beta\vec{t}) &= y_0(\alpha s + \beta t) + \sum_{n=1}^{\infty}(\alpha s_n + \beta t_n)y_k \\
&= y_0\alpha s + y_0\beta t + \alpha\sum_{k=1}^{\infty} s_k y_k + \beta\sum_{k=1}^{\infty} t_k y_k \\
&= \alpha\left(y_0 s + \sum_{k=1}^{\infty} s_k y_k\right) + \beta\left(y_0 t + \sum_{k=1}^{\infty} t_k y_k\right) \\
&= \alpha\Phi(\vec{s}) + \beta\Phi(\vec{t}).
\end{aligned}$$

This shows linearity. Note that the sums in the above calculations are valid since c has bounded sequences and l_1 has absolutely convergent sequences (which implies convergence). To show that Φ is bounded, we note that, from Real Analysis, for convergent sequences, $\left|\lim_{n\to\infty} x_n\right| \leq ||\vec{x}||$, where, $||\vec{x}|| = \sup_{k\in\mathbb{N}} |x_k|$. So, using this and the first inequality in the proof of this theorem, we get,

$$|\Phi(\vec{x})| = ||\vec{x}|| \sum_{k=0}^{\infty} |y_k| = ||\vec{x}||\ ||\vec{y}||.$$

This shows Φ is bounded. We note that $\vec{e}_k \to 0$ for all $k \in \mathbb{N}$ and so, $\Phi(\vec{e}_k) = 0\cdot y_0 + y_k \cdot 1 = y_k$. Next, as $\vec{e}_0 \to 1$, we have, $\Phi(\vec{e}_0) = 1 \cdot y_0 + \sum_{k=1}^{\infty} 1 \cdot y_k = y_0 + \sum_{k=1}^{\infty} \Phi(\vec{e}_k)$ and hence, $y_0 = \Phi(\vec{e}_0) - \sum_{k=1}^{\infty} \Phi(\vec{e}_k)$. This completes the proof. □

3.3 Some notes for c_0 and c_F

We denote the space of all complex sequences that converge to 0 (that is, space of null sequences) by c_0, and the space of complex sequences having at most finitely many non-zero terms (that is, space of sequences of finite support) by c_F. Clearly, $c_F \subseteq c_0 \subseteq c$ (verification left as an easy exercise). Now, for every null sequence, from Th.8, we can have a unique representation $\vec{x} = \sum_{k=1}^{\infty} x_k \vec{e}_k$. So, the Schauder basis for l_p spaces is a Schauder basis for c_0 as well. Now, from Th.9, if we replace $(y_k)_{k=0}^{\infty}$ by $(y_k)_{k=1}^{\infty}$ (just for convenience) and consider only null sequences, then $l = 0$, in the function of Th.9. If we restrict Φ to c_0, we can find a bounded linear functional $\Phi|_{c_0}$ on c_0 such that $\Phi|_{c_0}(\vec{e}_k) = y_k$. Similarly, as c_F is a subspace of c_0, and as all of $\vec{e}_k$ are in c_F, it is a Schauder basis for c_F as well. Restricting $\Phi|_{c_0}$ to c_F, we can find a bounded linear functional $\Phi|_{c_0}$ on c_0 such that $\left.\Phi|_{c_0}\right|_{c_F}(\vec{e}_k) = y_k$. For convenience, we denote each of these restrictions by Φ without any ambiguity.

4 Main Theorems

4.1 Remark

The following theorems establish the isomorphisms between a normed space and the dual of another normed space. We say that ***the dual of*** X ***is*** Y if and only if ***the dual of*** X ***is isomorphic to*** Y.

4.2 Theorems

Theorem 10. *Dual of* $\mathbb{R}^n$ *is* $\mathbb{R}^n$.

Proof. Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and let $f \in (\mathbb{R}^n)' = (\mathbb{R}^n)^*$. Consider the standard basis $\{e_1, e_2, \cdots e_n\}$ of $\mathbb{R}^n$. Then, we must have, $x = \sum_{k=1}^{n} x_k e_k$. As f is linear, we shall get, $f(x) = \sum_{k=1}^{n} x_k f(e_k)$. Let us define $T : (\mathbb{R}^n)' \to \mathbb{R}^n$ by $T(f) = (f(e_1), f(e_2), \cdots, f(e_n))$.

- <u>T is well-defined.</u>

  As f is a functional, it is obviously a map and so, each of $f(e_k)$ are unique. Hence, to f, T assigns a unique element of $\mathbb{R}^n$. So, T is well defined.

- <u>T is one to one.</u>

  Let f and g be two elements in the domain of T such that $T(f) = T(g)$. Then, $f(e_k) = g(e_k)$, for every $k = 1, 2, \cdots, n$. Since the e_k's constitute a basis, from Th.1, $f = g$, since $\mathbb{R}^n$ is finite dimensional. So, T is one to one.

- T is onto.

  Let $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. Define $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \sum_{k=1}^{n} x_k y_k$. Then, it is a linear functional on $\mathbb{R}^n$ (which means $h \in (\mathbb{R}^n)'$) such that $y_j = h(e_j)$ for every $j = 1, 2, \cdots, n$, as per Th.4. Here, y was arbitrary. So, if we vary y, it is clear that, for each of them, we can find a map as given by h. Furthermore, $y_j = h(e_j)$ means, $(y_1, y_2, \cdots, y_n) = (h(e_1), h(e_2), \cdots, h(e_n)) = T(h)$, so, to each y, we can find an h in $\mathbb{R}^n$ such that $y = T(h)$. This means T is onto.

- T is linear.
  Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in (\mathbb{R}^n)'$. Then,

  $$T(\alpha f + \beta g) = ((\alpha f + \beta g)(\vec{e}_1), (\alpha f + \beta g)(\vec{e}_2), \cdots, (\alpha f + \beta g)(\vec{e}_n))$$

  which, on simplification, gives

  $$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$

  as per the usual rule of addition and scalar multiplication for $\mathbb{R}^n$. So, T is linear.

- T preserves norm.
  Let $x \neq 0 \in \mathbb{R}^n$. Then, by Cauchy-Schwarz inequality,

  $$|f(x)| \leq \sum_{k=1}^{n} |x_k u_k| \leq \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |u_k|^2\right)^{1/2} = ||x|| \left(\sum_{k=1}^{n} |u_k|^2\right)^{1/2} = ||x||\ ||u||$$

  where, $u = (u_1, u_2 \cdots u_n) = (f(e_1), f(e_2), \cdots, f(e_n)) = T(f)$. This gives,

  $$\frac{|f(x)|}{||x||} \leq ||u|| \ \Rightarrow\ \sup_{x \neq 0} \frac{|f(x)|}{||x||} \leq ||u|| \ \Rightarrow\ ||f|| \leq ||u||.$$

  On the other hand, by definition of $||f||$, we have, for $x \neq 0$, we have,

  $$||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||} \geq \frac{|f(x)|}{||x||}$$

  and so, taking, in particular, $x = u$, we get,

  $$||f|| \geq ||u||$$

  since, $|f(u)| = \left|\sum_{k=1}^{n} u_k \cdot u_k\right| = \sum_{k=1}^{n} u_k^2 = ||u||^2$. The last two inequalities give $||f|| = ||T(f)||$. Note that we are allowed to do all these if $u \neq 0$. If $u = 0$, then it shall mean $f(e_1) = f(e_2) = \cdots = f(e_n) = 0$. So, we get $f(x) = 0$ for all x and so, f becomes a zero function. So, $||f|| = 0$. As $f = \mathcal{O}$, $||T(f)|| = ||T(\mathcal{O})|| = ||\mathcal{O}|| = 0 = ||f||$. Hence, the norm is preserved.

These show that $(\mathbb{R}^n)'$ is isomorphic to $\mathbb{R}^n$, that is, dual of $\mathbb{R}^n$ is $\mathbb{R}^n$. $\square$

Theorem 11. *Dual of l_1 is l_∞.*

Proof. Let $(\vec{e}_k)_{k=1}^{\infty}$ be a sequence in l_1 such that $\vec{e}_k = (\delta_{jk})_{j=1}^{\infty}$. Then, this sequence is a Schauder basis for l_1, from Th.5. Let $\vec{x} = (x_1, x_2, \cdots) \in l_1$ and $f \in l_1'$. Then, from Th.5, we have $\vec{x} = \sum_{k=1}^{\infty} x_k \vec{e}_k$ and from Th.2, as f is a bounded linear operator, we have, $f(\vec{x}) = \sum_{k=1}^{\infty} x_k f(\vec{e}_k)$. Let us write $\gamma_k = f(\vec{e}_k)$. Now, as f is a bounded linear functional, $|\gamma_k| = |f(\vec{e}_k)| \leq ||f||\ ||\vec{e}_k||$. By definition of $\vec{e}_k$, it is clear that one of its terms is 1 and the rest are 0. So, under l_1 norm, $||\vec{e}_k|| = 1$. Hence, we have,

$$|\gamma_k| = |f(\vec{e}_k)| \leq ||f|| \ \Rightarrow \ \sup_{k\in\mathbb{N}} |f(\vec{e}_k)| \leq ||f|| < \infty \tag{1}$$

which shows $(f(\vec{e}_k))_{k=1}^{\infty} = (\gamma)_{k=1}^{\infty} \in l_\infty$. Let us define $T : l_1' \to l_\infty$ by $T(f) = (f(\vec{e}_k))_{k=1}^{\infty}$. Then,

- <u>T is well defined.</u>

  Here, f is a functional, so, is a function. Hence, it assigns a unique value $f(\vec{e}_k)$ to each $\vec{e}_k$. Hence, each of the elements of the sequence in the definition of T are unique, for each given f. Thus, T is well defined.

- <u>T is one to one.</u>

  Let f and g be two elements of l_1' such that $T(f) = T(g)$. Then, $f(\vec{e}_k) = g(\vec{e}_k)$ for every $k \in \mathbb{N}$. Hence, from Th.3, $f = g$, making T one to one.

- <u>T is onto.</u>

  Let $\vec{y} = (y_n)_{n=1}^{\infty}$ be any element in l_∞. Let $h : l_1 \to \mathbb{K}$ be defined by $h(\vec{x}) = \sum_{k=1}^{\infty} x_k y_k$. Then, from Th.7, it follows that h is a bounded linear functional on l_1 such that $y_j = h(\vec{e}_j)$. This means $h \in l_1$ is such that $\vec{y} = T(h)$. As $\vec{y}$ was arbitrary, we may conclude that for each $\vec{y} \in l_\infty$, we can find $h \in l_1'$ such that $\vec{y} = T(h)$. Hence T is onto.

- <u>T is linear.</u>

  Let $\alpha, \beta \in \mathbb{K}$ and $f, g \in l_1'$. Then,

  $$T(\alpha f + \beta g) = ((\alpha f + \beta g)(\vec{e}_k))_{k=1}^{\infty}$$

  which, on simplification, gives

  $$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$

  as per the usual rule of addition and scalar multiplication for complex sequences. So, T is linear.

- <u>T preserves norm.</u>

  From (1), it is clear that $||T(f)|| \leq ||f||$. For the reversed inequality, we note that

  $$\sum_{k=1}^{\infty} |x_k \gamma_k| \leq \sup_{k\in\mathbb{N}} |\gamma_k| \sum_{k=1}^{\infty} |x_k| = ||\vec{x}||\ ||T(f)||$$

and hence due to this absolute convergence, we can say that $|f(\vec{x})| \leq ||\vec{x}||\ ||T(f)||$. Thus, for $\vec{x} \neq 0$, this gives

$$\frac{|f(\vec{x})|}{||\vec{x}||} \leq ||T(f)|| \ \Rightarrow\ \sup_{\vec{x}\neq 0} \frac{|f(\vec{x})|}{||\vec{x}||} \leq ||T(f)|| \ \Rightarrow\ ||f|| \leq ||T(f)||.$$

Hence T preserves norm, as the last two inequalities give $||f|| = ||T(f)||$.

Hence l_1' is isomorphic to l_∞, that is, dual of l_1 is l_∞. □

Theorem 12. *Let $p, q > 1$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of l_p is l_q.*

Proof. Let $\vec{x} = (x_n)_{n=1}^\infty \in l_p$ and $f \in l_p'$. Let $(\vec{e}_k)_{k=1}^\infty$ be a Schauder basis of l_p as given by Th.5. Then, we can write $\vec{x} = \sum_{k=1}^\infty x_k \vec{e}_k$. So, from Th.3, $f(\vec{x}) = \sum_{k=1}^\infty x_k f(\vec{e}_k)$. Let us write $\gamma_k = f(\vec{e}_k)$. Then $f(\vec{x}) = \sum_{k=1}^\infty x_k \gamma_k$. Let $n \in \mathbb{N}$. Define a sequence $\hat{\vec{x}} = (\hat{x}_k)_{k=1}^\infty$ as

$$\hat{x}_k = \begin{cases} \dfrac{|\gamma_k|^q}{\gamma_k} & \text{if } \gamma_k \neq 0 \text{ and } 1 \leq k \leq n \\ 0 & \text{if } \gamma_k = 0 \text{ or } k > n. \end{cases}$$

Then,

$$\sum_{k=1}^\infty |\hat{x}_k|^p = \sum_{k=1}^n \left(\frac{|\gamma_k|^q}{|\gamma_k|}\right)^p = \sum_{k=1}^n |\gamma_k|^{p(q-1)} = \sum_{k=1}^n |\gamma_k|^q < \infty$$

being a finite sum and as $\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow \frac{1}{p} = 1 - \frac{1}{q} \Leftrightarrow \frac{q}{p} = q - 1 \Leftrightarrow q = p(q-1)$. So, $\hat{\vec{x}} \in l_p$.

Hence, we obtain,

$$f(\hat{\vec{x}}) = \sum_{k=1}^\infty \hat{x}_k \gamma_k = \sum_{k=1}^n \frac{|\gamma_k|^q}{\gamma_k} \cdot \gamma_k = \sum_{k=1}^n |\gamma_k|^q > 0.$$

Note that in the finite sum involving γ_k, it is not necessary that all of the n terms are non-zero. We have assumed that the sum is non-zero, which makes sure that one of the terms (at least) isn't zero. The cancellations are being performed only for such terms. We have not stated the condition of considering non-zero terms among the first n terms is not mentioned in the sigma notation for the mere sake of convenience.

Now, due to boundedness of f, we have,

$$|f(\hat{\vec{x}})| = f(\hat{\vec{x}}) \leq ||f||\ ||\hat{\vec{x}}||.$$

Using the values of $||\hat{\vec{x}}||$ and $|f(\hat{\vec{x}})|$ in the above inequality, we get,

$$\sum_{k=1}^n |\gamma_k|^q \leq ||f|| \left(\sum_{k=1}^n |\gamma_k|^q\right)^{1/p} \quad \Rightarrow \quad \left(\sum_{k=1}^n |\gamma_k|^q\right)^{1/q} \leq ||f|| \quad \left(\because \frac{1}{q} = 1 - \frac{1}{p}\right).$$

We know that a real valued function on $\mathbb{R}$ given by $f(t) = t^{1/q}$ for $t \geq 0$ is continuous and in the last inequality, $n \in \mathbb{N}$ is arbitrary. As the sum in the inequality represents the sequence of partial

sums of a non-negative sequence, the resulting sequence of partial sums is non-decreasing and is bounded. Hence, taking limits as $n \to \infty$, we get,

$$\left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q} \leq ||f|| < \infty.$$

If we consider the case when the sum is zero, then, by definition, it must be that all the first n terms of the $(\gamma_k)_{k=1}^{\infty}$ are zeroes and so $\left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q} = 0 \leq ||f||$. Hence, in all possible cases, we have,

$$\left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q} \leq ||f|| < \infty$$

and so, $(\gamma_k)_{k=1}^{\infty} = (f(\vec{e}_k))_{k=1}^{\infty} \in l_q$.
Now, define $T : l_p' \to l_q$ by $T(f) = (f(\vec{e}_k))_{k=1}^{\infty}$. Then,

- T is well defined.

  We have already shown that $T(f) \in l_p$ and as f is a functional, it assigns a unique value to each of $\vec{e}_k$ and hence the sequence in the definition of T is unique. Hence, T is well defined.

- T is one to one.

  Let f and g be two elements of l_1' such that $T(f) = T(g)$. Then, $f(\vec{e}_k) = g(\vec{e}_k)$ for every $k \in \mathbb{N}$. Hence, from Th.3, $f = g$, making T one to one.

- T is onto.

  Let $\vec{y} = (y_n)_{n=1}^{\infty} \in l_q$. Then, from Th.6, it is clear that the map $h : l_q \to \mathbb{K}$ defined by $h(\vec{x}) = \sum_{k=1}^{\infty} x_k y_k$ is a bounded linear functional such that $y_j = h(\vec{e}_j)$. So, varying y, we get different such h and each h is in l_p'. Hence, to each $\vec{y} \in l_q$ there corresponds an $h \in l_p'$ such that $T(h) = \vec{y}$. Hence T is onto.

- T is linear.

  Let $\alpha, \beta \in \mathbb{K}$ and $f, g \in l_1'$. Then,

  $$T(\alpha f + \beta g) = ((\alpha f + \beta g)(\vec{e}_k))_{k=1}^{\infty}$$

  which, on simplification, gives

  $$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$

  as per the usual rule of addition and scalar multiplication for complex sequences. So, T is linear.

- T preserves norm.

  From previous discussions, we already have that $||T(f)|| \leq ||f||$. For the reverse inequality, we can observe that, by an application of Hölder's inequality, we may obtain

  $$\sum_{k=1}^{\infty} |x_k \gamma_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q} = ||\vec{x}||\ ||(f(\vec{e}_k))_{k=1}^{\infty}|| = ||\vec{x}||\ ||T(f)||.$$

Due to this absolute convergence, we can say that $|f(\vec{x})| \leq ||\vec{x}||\ ||T(f)||$. Thus, for $\vec{x} \neq 0$, this gives

$$\frac{|f(\vec{x})|}{||\vec{x}||} \leq ||T(f)|| \ \Rightarrow \ \sup_{\vec{x}\neq 0} \frac{|f(\vec{x})|}{||\vec{x}||} \leq ||T(f)|| \ \Rightarrow \ ||f|| \leq ||T(f)||.$$

Hence T preserves norm, as the last two inequalities give $||f|| = ||T(f)||$.

This shows us that l_p' is isomorphic to l_q, that is, the dual of l_p is l_q. □

<u>Note</u> : If we merely interchange the role of p and q in the above theorem, we can show that the dual of l_q is l_p. It is left as an exercise for the reader.

Theorem 13. *Dual of c is l_1.*

Proof. Let $\vec{x} = (x_n)_{n=1}^{\infty} \in c$. Consider a Schauder basis for c as in Th.8. Let $f \in c'$. Taking the representation and notations in Th.8 and as f is a bounded linear operator on c, we can write

$$f(\vec{x}) = l f(\vec{e}_0) + \sum_{n=1}^{\infty} (x_k - l) f(\vec{e}_k).$$

Let $n \in \mathbb{N}$. Define a sequence $\vec{h} = (h_k)_{k=1}^{\infty}$ by

$$h_k = \begin{cases} \operatorname{sgn}(f(\vec{e}_k)) & \text{if } 1 \leq k \leq n \\ \\ 0 & \text{if } k > n. \end{cases}$$

As $\vec{h}$ has finitely many non-zero terms, it converges to zero, so $l = 0$ for $\vec{h}$. Now, $\left|\left|\vec{h}\right|\right| = \sup_{k\in\mathbb{N}} |h_k| = \sup\{0,\ |\operatorname{sgn}(f(\vec{e}_1))|,\ |\operatorname{sgn}(f(\vec{e}_2))|, \cdots, |\operatorname{sgn}\ (f(\vec{e}_n))|\}$. If $\left|\left|\vec{h}\right|\right| = 0$, then by definition of $\vec{h}$ and the signum function, it follows that $f(\vec{e}_k) = 0$ for $1 \leq k \leq n$. This means, $0 = \sum_{k=1}^{n} |f(\vec{e}_k)| \leq ||f||$ and hence, as n is arbitrary, in this case $\sum_{k=1}^{\infty} |f(\vec{e}_k)| \leq ||f||$. If $\left|\left|\vec{h}\right|\right| \neq 0$, at least one of the signum functional values are non-zero. By definition of signum functions, it follows that for the non-zero signum functional values, say $\operatorname{sgn}(f(\vec{e}_k))$, we have

$$|\operatorname{sgn}(f(\vec{e}_k))| = \frac{|(f(\vec{e}_k))|}{|(f(\vec{e}_k))|} = 1.$$

And so, $\left|\left|\vec{h}\right|\right| = \sup\{0, 1\} = 1$. So, we have, in this case, as $\vec{h} \in c$ and f is bounded,

$$\left|f(\vec{h})\right| = \left|0 \cdot \vec{e}_0 + \sum_{k=1}^{n} \operatorname{sgn}(f(\vec{e}_k)) f(\vec{e}_k)\right| = \left|\sum_{k=1}^{n} \operatorname{sgn}(f(\vec{e}_k)) f(\vec{e}_k)\right| = \sum_{k=1}^{n} |f(\vec{e}_k)| \leq \left|\left|\vec{h}\right|\right| ||f|| = ||f||.$$

As n was arbitrary, we get, $\sum_{k=1}^{\infty} |f(\vec{e}_k)| \leq ||f||$. This shows absolute convergence of the series $\sum_{k=1}^{\infty} f(\vec{e}_k)$ and hence, the series itself is convergent. So, we have $\sum_{k=0}^{\infty} |a_k| < \infty$, where, $a_k = f(\vec{e}_k)$

for $k \in \mathbb{N}$, and, $a_0 = f(\vec{e}_0) - \sum_{k=1}^{\infty} |a_k| = f(\vec{e}_0) - \sum_{k=1}^{\infty} |f(\vec{e}_k)|$. Now, define $T : c' \to l_1$ by $T(f) = \vec{a} = (a_k)_{k=0}^{\infty} = \left(f(\vec{e}_0) - \sum_{k=1}^{\infty} |f(\vec{e}_k)|, f(\vec{e}_1), f(\vec{e}_2), \cdots \right)$. Then,

- <u>T is well defined.</u>
  As f assigns a unique value to each of $\vec{e}_k$ $(k \in \mathbb{N} \cup \{0\})$, T is well defined.

- <u>T is one to one.</u>

  Let $f, g \in c'$ such that $T(f) = T(g)$. Then, $f(\vec{e_k}) = g(\vec{e_k})$ for $k \in \mathbb{N}$ and so, $\sum_{k=1}^{\infty} f(\vec{e}_k) = \sum_{k=1}^{\infty} g(\vec{e}_k)$. We also have, $f(\vec{e}_0) - \sum_{k=1}^{\infty} |f(\vec{e}_k)| = g(\vec{e}_0) - \sum_{k=1}^{\infty} |g(\vec{e}_k)|$ and so, $f(\vec{e}_0) = g(\vec{e}_0)$. Hence, from Th.3, $f = g$. This shows T is one to one.

- <u>T is onto.</u>
  Let $\vec{y} = (y_n)_{n=1}^{\infty} \in l_1$. Define $\Phi : c \to \mathbb{K}$ by

  $$\Phi(\vec{x}) = y_0 \cdot l + \sum_{k=1}^{\infty} x_k y_k.$$

  Then, from Th.9, Φ is a bounded linear functional on c such that $\Phi(\vec{e}_k) = y_k$ for all $k \in \mathbb{N}$ and $\Phi(\vec{e}_0) - \sum_{k=1}^{\infty} \Phi(\vec{e}_k) = y_0$. So, varying $\vec{y}$, we get such Φ for each of them. So, to each $\vec{y}$, we can find a map Φ in c', such that $T(\Phi) = \vec{y}$, as per our definition of T. Hence, T is onto.

- <u>T is linear.</u>
  Let $\alpha, \beta \in \mathbb{K}$ and $\Psi, \zeta \in c'$. Then, we will get $T(\alpha\Psi + \beta\zeta) = \alpha T(\Psi) + \beta T(\zeta)$. We have to arrange the terms in the first component and the rest is similar to what we have done in the proofs of previous theorems. (The details are left as an exercise.)

- <u>T preserves norm.</u>

  As per our considerations, we can write $f(\vec{x}) = a_0 l + \sum_{k=1}^{\infty} x_k a_k$. As in Th.9, we have, $|l| \leq ||\vec{x}||$. So, we get $f(\vec{x}) \leq \left(|a_0| + \sum_{k=1}^{\infty} |a_k| \right) ||\vec{x}|| = \left(\sum_{k=0}^{\infty} |a_k| \right) ||\vec{x}||$. Proceeding as in the proofs of previous theorems, $||f|| \leq \sum_{k=0}^{\infty} |a_k| = ||\vec{a}||$.

  To obtain the reversed inequality, let $n \in \mathbb{N}$. Define a sequence $\vec{p} = (p_k)_{k=1}^{\infty}$ as

  $$p_k = \begin{cases} \operatorname{sgn}(a_k) & \text{if } 1 \leq k \leq n \\ \\ \operatorname{sgn}(a_0) & \text{if } k > n. \end{cases}$$

Since $p_k = \text{sgn}(a_0)$ for all except finitely many k, $p_k \to \text{sgn}(a_0)$. So, we can write,

$$f(\vec{p}) = a_0\text{sgn}(a_0) + \sum_{k=1}^{n} \text{sgn}(a_k)a_k + \sum_{k=n+1}^{\infty} \text{sgn}(a_0)a_k = |a_0| + \sum_{k=1}^{n} |a_k| + \sum_{k=n+1}^{\infty} \text{sgn}(a_0)a_k.$$

So, we have,

$$f(\vec{p}) = \left| |a_0| + \sum_{k=1}^{n} |a_k| + \text{sgn}(a_0) \sum_{k=n+1}^{\infty} a_k \right| \leq ||f||\ ||\vec{p}||.$$

Taking limits as $n \to \infty$, the first sum is convergent as the sequence is in l_1 and the second sum converges to zero as it is the remainder of the first sum. So, we get,

$$\sum_{k=0}^{\infty} |a_k| = ||\vec{a}|| \leq ||f||\ ||\vec{p}||.$$

Due to the presence of signum function in the definition of $\vec{p}$,as before, $||\vec{p}|| = 1$, if at least one of the signum functions isn't zero. So, in such a case, $||\vec{a}|| \leq ||f||$. Hence we get $||f|| = ||\vec{a}||$. If all of the signum functions are zeroes, then $||\vec{p}|| = 0$, and so $||f|| \leq 0$ (first inequality under this heading), giving us $||f|| = 0$. Note that all of the said signum functions being zero means for arbitrary $n \in \mathbb{N}$, $a_n = 0$, so the image is a zero sequence. So, $||f|| = ||\vec{a}||$. Hence, in all cases, $||f|| = ||T(f)||$. So, T preserves norm.

Thus, dual of c is isomorphic to l_1, that is, dual of c is l_1. □

4.3 Dual spaces of c_0 and c_F

Here we shall discuss, in brief, how the duals of c_0 and c_F also happen to be l_1. Complete details are left as exercises for the reader. Notations are used as before. From sec.3.3, it is clear that for $\vec{x} \in c_0$, we have $f(\vec{x}) = \sum_{k=1}^{\infty} x_k f(\vec{e}_k)$. Now, define $\vec{h}$ as in the proof of Th.13. It is a null sequence. Considering the various cases for its norm and proceeding similarly, we get $\sum_{k=1}^{\infty} |f(\vec{e}_k)| \leq ||f|| < \infty$. So the sequence $(f(\vec{e}_k))_{k=1}^{\infty}$ is in l_1. Define $T : c_0' \to l_1$ by $T(f) = (f(\vec{e}_k))_{k=1}^{\infty}$. T being well defined and one to one follows as before. Sec.3.3 also helps us to show that it is onto. To show the norm preserving property do as in Th.11, with slight alterations wherever necessary. For c_F, give this proof and you can get away with it, since $\vec{h}$ is of finite support as well.

5 Final Discussion - What is the dual space of the sequence space l_∞?

One question shall arise - isn't dual of l_∞ isomorphic to l_1? One might see this as an obvious result, but sadly, this is not the case. Note that we have found isomorphic spaces for the dual spaces of all "good" sequence spaces, except l_∞. So, why is it so? This is because l_1 is a separable space and l_∞ is not. As l_∞ is not separable, its dual cannot be linearly isometric, that is, isomorphic to l_1, a separable space. See [6] for more information. See [9] for how the dual is obtained. It requires some general concepts of Measure Theory and Radon measure. So, we shall not go beyond this with the discussion. We can also find isomorphic spaces for function spaces. See [1], [5], [6], [7] and [8] for such examples.

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